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The Dirac oscillator with a Coulomb-like tensor potential

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Abstract
We study the Dirac equation with a tensor potential which contains a term linear in r and a Coulomb-like term. The eigenstates and eigenvalues are obtained exactly. We found that the energy spectrum and the degeneracy of the levels depend on the alignment of spin with the orbital angular momentum. For parallel alignment, the second term in the potential makes no contribution to the energy levels.

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The Schrödinger equation with an oscillator potential and a spin–orbit coupling was an important tool for the nuclear shell model. Therefore, it is important to study its extension to the Dirac equation. The relativistic harmonic oscillator has also been used for quarks with applications in the meson and baryon spectroscopy. A special-type harmonic oscillator potential is achieved by replacing the linear momentum operator $\vec{p}$, in the Dirac equation, with $\vec{p} - \mathrm{i}\hat{r}\beta U(r)$. Here, α_i and β are the usual Dirac matrices and $\hat{r} = \vec{r}/r$. When U is a linear function of r, one obtains an oscillator with a spin–orbit coupling term. This is called the Dirac oscillator [1]. Such an oscillator has a strong spin–orbit term and an infinite degeneracy [2]. Another way of introducing a harmonic potential in the Dirac equation is achieved by mixing vector and scalar harmonic potentials. This gives a normal oscillator without the spin–orbit coupling [3]. Kukulin *et al* [4] have combined these two approaches and obtained an oscillator with independent couplings for the central and spin–orbit parts. Recently, the Dirac Hamiltonian with harmonic oscillator potentials is used to explain the so-called pseudo-spin symmetry in nuclear interactions [5, 6]. In [7, 8], we give some recent publications on the application of the Dirac oscillator.

In this study, we consider the potential U as a sum of two terms: a term linear in r and a Coulomb-like term. We show that, with this choice, the Dirac equation can be solved exactly. We obtain the energy spectrum and the corresponding wavefunction using the so-called Nikiforov–Uvarov (NU) method. We compare our results with the known spectrum of the Dirac oscillator. The contribution of the Coulomb-like term is analysed.

The time-independent Dirac equation [9] with energy E can be written as

$$H\psi = E\psi \tag{1}$$

where

$$H = \vec{\alpha}\cdot\vec{p} + m\beta + \phi$$

is the Dirac Hamiltonian in the presence of a potential ϕ. In general, the potential ϕ may be written as a sum of a vector, a scalar and a tensor potential in the form

$$\phi = V(r) + \beta S(r) + \mathrm{i}\beta\vec{\alpha}\cdot\hat{r}U(r). \tag{2}$$

The Dirac equation is already studied for different choices of V, S and U [5, 6, 9, 12]. Here, we will assume that the vector and scalar potentials are zero and U is given as

$$U(r) = m\omega\cdot r - \frac{\alpha}{r}. \tag{3}$$

The wavefunction satisfying equation (1) can be written as

$$\psi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \tag{4}$$

where ϕ_1 and ϕ_2 are the large and small components, respectively. From equation (1), it follows that these components satisfy

$$(E-m)\phi_1 = \vec{\sigma}\cdot(\vec{p} + \mathrm{i}\hat{r}U)\phi_2 \tag{5}$$

and

$$(E+m)\phi_2 = \vec{\sigma}\cdot(\vec{p} - \mathrm{i}\hat{r}U)\phi_1 \tag{6}$$

where σ_i are the Pauli matrices. Multiplying equation (5) by $(E+m)$ and using the resulting equation in equation (6), one obtains

$$(E^2-m^2)\phi_1 = [\vec{\sigma}\cdot(\vec{p} + \mathrm{i}\hat{r}U)][\vec{\sigma}\cdot(\vec{p} - \mathrm{i}\hat{r}U)]\phi_1. \tag{7}$$

Using the properties of Pauli spin matrices, this equation can be expressed as

$$(E^2-m^2)\phi_1 = [(\vec{p} + \mathrm{i}\hat{r}U)\cdot(\vec{p} - \mathrm{i}\hat{r}U) + \mathrm{i}\vec{\sigma}\cdot((\vec{p} + \mathrm{i}\hat{r}U)\times(\vec{p} - \mathrm{i}\hat{r}U))]\phi_1. \tag{8}$$

Employing the commutation relations of the operators involved and transforming to the spherical coordinates, the last equation can be written as

$$(E^2-m^2)\phi_1 = \left[\vec{p}^{\,2} + U^2 - 2\left(\frac{U}{r}\right) - \left(\frac{\mathrm{d}U}{\mathrm{d}r}\right) - 4\left(\frac{U}{r}\right)(\vec{L}\cdot\vec{S})\right]\phi_1 \tag{9}$$

where

$$\vec{L} = \vec{r}\times\vec{p}, \qquad \vec{S} = \tfrac{1}{2}\vec{\sigma}.$$

If we take only the linear term in U, equation (9) reduces to

$$(E^2-m^2)\phi_1 = [\vec{p}^{\,2} + m^2\omega^2r^2 - 3m\omega - 4m\omega(\vec{L}\cdot\vec{S})]\phi_1. \tag{10}$$

For the nonrelativistic limit, we define $\varepsilon = E - m$ and use the relation $\varepsilon \ll m$ to write $(E^2 - m^2)$ as $2m\varepsilon$. In this limit, equation (10) becomes

$$\varepsilon\phi_1 = \frac{1}{2m}[\vec{p}^{\,2} + m^2\omega^2r^2 - 3m\omega - 4m\omega(\vec{L}\cdot\vec{S})]\phi_1. \tag{11}$$

This represents a harmonic oscillator with a spin–orbit coupling term. This was called the Dirac oscillator [1]. Let us return to equation (10) and express the operator $\vec{L}\cdot\vec{S}$ as

$$\vec{L}\cdot\vec{S} = \tfrac{1}{2}(\vec{J}^2 - L^2 - \vec{S}^2) \tag{12}$$

where $\vec{J}^2$, $\vec{L}^2$ and $\vec{S}^2$ are, respectively, the squares of the total, orbital and spin angular momentum operators. Taking into account the commutation relations of the operators in equation (10), it is more convenient to express the two-component wavefunction ϕ_1 in the form

$$\frac{\chi(r)}{r}\varphi_{\{l\frac{1}{2}\}\{jm\}} \tag{13}$$

where $\varphi_{\{l\frac{1}{2}\}\{jm\}}$ are the spinor-spherical harmonics. They are constructed by coupling the two-dimensional spinors η_ν with the spherical harmonics $Y_{l\mu}$ as follows:

$$\varphi_{\{l\frac{1}{2}\}\{jm\}} = \sum_{\mu\nu}\left\langle l\mu, \frac{1}{2}\nu \middle| jm\right\rangle Y_{l\mu}\eta_\nu. \tag{14}$$

Replacing these into equation (10), we end up with the following second-order differential equation for the wavefunction χ:

$$(E^2 - m^2)\chi(r) = -\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2} + m^2\omega^2 r^2 + \frac{A}{r^2} - \frac{B}{m\omega}\right)\chi(r) \tag{15}$$

where

$$\begin{aligned} A &= \alpha\left[\alpha + 1 + 2\left(j(j+1) - l(l+1) - \frac{3}{4}\right)\right] \\ B &= 2m\omega\alpha + 3m\omega + 2m\omega\left[j(j+1) - l(l+1) - \frac{3}{4}\right]. \end{aligned} \tag{16}$$

It is convenient to introduce a dimensionless variable $s = m\omega r^2$ and transform equation (15) to the following form:

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}s^2} + \frac{1}{2s}\frac{\mathrm{d}\chi}{\mathrm{d}s} + \frac{1}{4s^2}[W - \Lambda s - s^2]\chi = 0 \tag{17}$$

where we have defined the following new parameters:

$$W = -l(l+1) - A, \qquad \Lambda = -\left[\frac{B + (E^2 - m^2)}{m\omega}\right]. \tag{18}$$

This equation has the form of a generalized equation of the hypergeometrical type [10]. Such equations can be expressed as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\tilde{\tau}(x)}{\sigma(x)}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\tilde{\sigma}(x)}{\sigma^2}y = 0 \tag{19}$$

where $\sigma(x)$ and $\tilde{\sigma}(x)$ are the polynomials, at most second degree, and $\tilde{\tau}$ is a first degree polynomial. To solve this equation, we will follow the NU method [10]. In summary, the method starts by taking $y = h(x)g(x)$ and choosing an appropriate h to obtain

$$\sigma(x)\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + \tau(x)\frac{\mathrm{d}g}{\mathrm{d}x} + \lambda g = 0 \tag{20}$$

which is referred as the equation of the hypergeometrical type. The first factor h is assumed to satisfy the following relations:

$$\frac{\mathrm{d}h(x)}{\mathrm{d}x} = \frac{\pi(x)}{\sigma(x)} \tag{21}$$

where

$$\pi = \frac{1}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x} - \tilde{\tau}\right) \pm \sqrt{\frac{1}{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x} - \tilde{\tau}\right)^2 - \tilde{\sigma} + k\sigma} \tag{22}$$

and

$$\tau = \tilde{\tau} + 2\pi. \tag{23}$$

The constant λ is defined by the relation

$$\lambda = k + \frac{\mathrm{d}\pi}{\mathrm{d}x}. \tag{24}$$

Here π is a polynomial and the parameter k is fixed to satisfy this condition. This means that the expression under the square root must be the square of a polynomial. It is known that the polynomial solutions of equation (20) are given by the Rodriguez formula:

$$g_n(x) = \frac{B_n}{\rho(x)} \frac{\mathrm{d}^n}{\mathrm{d}^n x}[\sigma^n(x)\rho(x)] \tag{25}$$

where B_n is a constant and the weight function ρ is determined by the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sigma\rho) = \tau\rho. \tag{26}$$

In order to obtain the polynomial solutions given by the Rodriguez formula, one has to impose the following condition:

$$\lambda_n = -n\frac{\mathrm{d}\tau}{\mathrm{d}x} - \frac{n(n-1)}{2}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x^2}, \qquad n = 0, 1, 2, 3, \ldots. \tag{27}$$

The details of this short summary can be found in [10].

Now we go back to equation (17) and apply the NU method to solve this equation. Comparing equations (17) and (19), we immediately see that

$$\tilde{\tau}(s) = 1, \qquad \sigma(s) = 2s, \qquad \tilde{\sigma}(s) = W - \Lambda s - s^2. \tag{28}$$

Substituting these polynomials into equation (25), we obtain π as

$$\pi = \frac{1}{2} \pm \frac{1}{2}[4s^2 + 4(\Lambda + 2k)s + 1 - 4W]^{\frac{1}{2}}. \tag{29}$$

According to the method given above, there are two values for k and thus four possibilities for π. After finding these four values, we choose the one which gives a τ function with negative derivative. This set of functions is

$$\begin{aligned} k &= -\tfrac{1}{2}(\Lambda + \sqrt{1-4W}) \\ \pi &= \tfrac{1}{2}(1 + \sqrt{1-4W} - s) \\ \tau &= 2 - 2s + \sqrt{1-4W}. \end{aligned} \tag{30}$$

Now we can combine these with equations (26) and (27) and obtain the eigenvalue equation:

$$2(2n+1) + \Lambda + \sqrt{1-4W} = 0. \tag{31}$$

Replacing the parameters Λ and W with the expressions given by equation (18), we arrive at the following formula for the energy values:

$$\frac{E^2 - m^2 + B}{m\omega} = 2(2n+1) + \sqrt{1 + 4l(l+1) + 4A}. \tag{32}$$

First, we look at $\alpha = 0$ limit. For this limit $A = 0$ and $B = 3m\omega + 2m\omega\big(j(j+1) - l(l+1) - \frac{3}{4}\big)$. There are two cases for the spin alignment, $j = l + \frac{1}{2}$ (aligned spin) and $j = l - \frac{1}{2}$ (unaligned spin). For these cases, equation (32) gives

$$\frac{(E_-^2 - m^2)}{m\omega} = 2(N - j) + 1, \qquad \text{for aligned spin} \tag{33}$$

$$\frac{(E_+^2 - m^2)}{m\omega} = 2(N + j) + 3, \qquad \text{for unaligned spin} \tag{34}$$

where N stands for $(2n + l)$. This is the well-known energy spectrum of the Dirac oscillator. As can be seen from these equations, there is an infinite degeneracy for the aligned spin case and a finite degeneracy for the unaligned spin case [1, 2].

We study equation (32) for a nonzero α for the two spin alignments. For unaligned spin case, using A and B from equation (16), we obtain

$$\frac{E_+^2 - m^2}{m\omega} = 2(N + j) + 3 - 4\alpha. \tag{35}$$

This coincides with equation (34) for $\alpha = 0$. For the aligned spin case, the energy values show an interesting behaviour. When A is replaced in the square root in equation (32), one obtains $1 + 4l(l + 1) + 4A = (2j + 2\alpha)^2$. The contribution of this term cancels with the contribution of $B/m\omega$ on the right-hand side. In fact, $B/m\omega$ reduces to $(2j + 2\alpha + 2)$ for the aligned spin case. This means the energy levels for the aligned spin case do not get any contribution from the α-dependent interaction term and the energy levels are given by the same formula as in equation (33).

The eigenfunctions are calculated using the Rodriguez formula. First, from equation (26), we solve ρ as

$$\rho(s) = \tfrac{1}{2}\,\mathrm{e}^{-s} s^{p} \tag{36}$$

where $p = \frac{1}{2}\sqrt{1 - 4W}$. Replacing σ and ρ in the Rodriguez formula, we find that the solutions $g(s)$ can be expressed in terms of the Laguerre polynomials as

$$g(s)_{np} = B_{np} L_n^p(s) \tag{37}$$

where B_{np} are some constants. For the first factor $\phi(s)$, we substitute $\pi(s)$ and $\sigma(s)$ into equation (21) and find that

$$h(s) = \mathrm{e}^{-\frac{1}{2}s} s^{\frac{1}{4}(1+2p)}. \tag{38}$$

Expressing these in terms of r, we find our wavefunctions:

$$\chi(r)_{np} = B_{np}(m\omega r^2)^{\frac{1}{4}(1+2p)} \exp\left(-\tfrac{1}{2}m\omega r^2\right) L_n^p(m\omega r^2). \tag{39}$$

At this point, one may look to the $\alpha = 0$ limit. In this limit, $p = \ell + \frac{1}{2}$, and one has [11]

$$L_n^{\ell+\frac{1}{2}}(m\omega r^2) = C_n^l\,{}_1\mathrm{F}_1\left(-n, \ell + \tfrac{3}{2}, m\omega r^2\right). \tag{40}$$

That is, we have the confluent hypergeometric functions or kummer functions in equation (39) and $\chi(r)$ coincides with the wavefunctions of the three-dimensional harmonic oscillator [12].

We could add to the Hamiltonian in equation (2) a term of the form $(I + \beta)V(r)$ where $V(r)$ is quadratic in r. This changes only the frequency of the central oscillator and the combined model is solvable exactly in an analytical manner.

We conclude that the Dirac equation with a tensor potential containing a linear and a Coulomb-like term is exactly solvable. The spectrum presents degeneracies. These are the same degeneracies that are observed in the spectrum of the Dirac oscillator. That is, the additional Coulomb-like tensor potential does not remove the degeneracies. The spectrum of the Dirac oscillator is explained by constructing a symmetry Lie algebra in [2]. Then, the Hamiltonian of the problem is related to the Casimir operators of the Lie algebra. This construction is far from trivial. Here, we can only make an observation that the Hamiltonian of the Dirac oscillator with the additional term has the same symmetry algebra.

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